

# Appendix

## A.1 Basic Notions

We take the terms *point*, *line*, *plane*, and *space* as undefined. We also use the concept of a *set* and a *subset*, *belongs to* or *is an element of* a set. In a formal axiomatic approach we use only the properties of the undefined terms stated in the axioms and are not allowed to make conclusions based on drawings. Our first axiom regards lines, planes and space as sets of points.

**Axiom A.1.** *Lines, planes and space are sets of points. Space contains all points.*

In geometry, it is common to use the terms *on*, *in*, *passes through*, and *lies on*. We say that a point is *on* a line rather than *belongs to* a line. Synonymously, we say that a line *passes through* a given point. We also say that a point is *on* a plane or *in* a plane. A line *lies in* a plane if it is a subset of the plane, i.e. if every point on the line is also in the plane. Points that are on the same line are called **collinear** and if they are on the same plane they are **coplanar**. The following axioms describe the fundamental relationships among points, lines and planes. The accompanied figures are only models for the relationships.

**Axiom A.2.** *Any two distinct points are on exactly one line. Every line contains at least two points.*

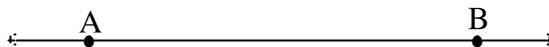
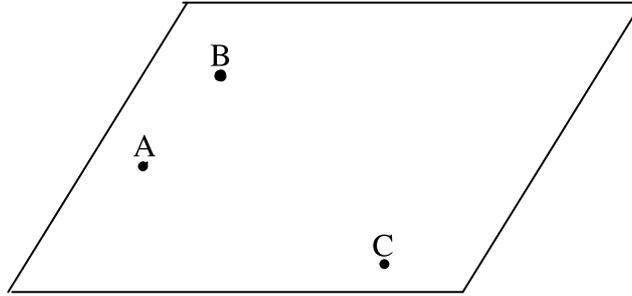


Figure 1

**Remark.** It may seem that this axiom implies that all lines are straight, otherwise more than one line could be drawn through two points. First notice that “straightness” has not

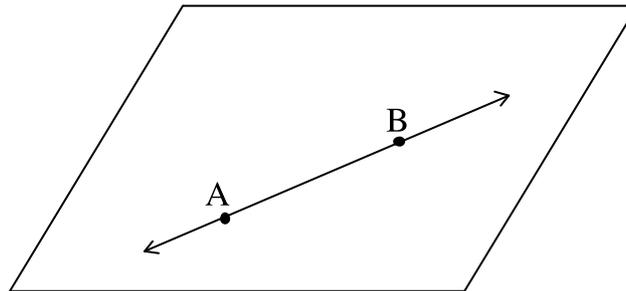
yet been defined. Second we will show in Chapter 5 an example of geometry in which lines are objects that satisfy Axiom A.2 and other axioms in this section but are not straight.

**Axiom A.3.** *Any three noncollinear points are on exactly one plane. Each plane contains at least three noncollinear points.*



**Figure 2**

**Axiom A.4.** *If two points of a line are in a plane, then the entire line is in the plane.*



**Figure 3**

**Axiom A.5.** *In space if two planes have a point in common, then the planes have an entire line in common.*

**Axiom A.6.** *In space if there exist at least four points that are noncoplanar.*

*Remarks on the Axioms:*

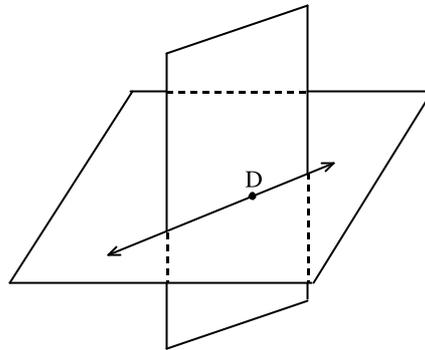


Figure 4

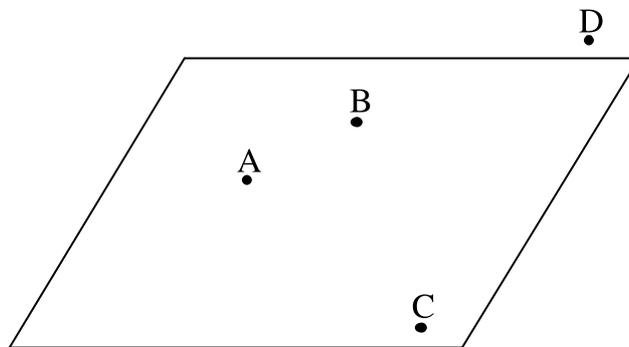


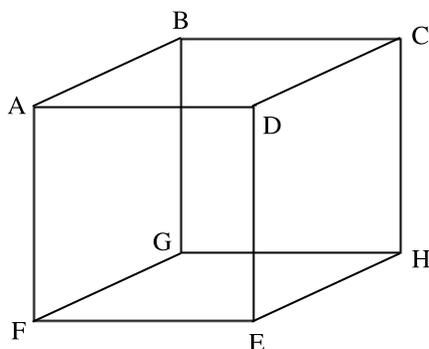
Figure 5

- Axiom A.2 is often encountered in an equivalent form: *Two points determine a unique line* or *There is one and only one line passing through two distinct points.*
- Similarly, an equivalent form of Axiom A.3 is: *Three noncollinear points determine a unique plane.*
- Notice the second sentence in each of the first two axioms. We know intuitively that lines and planes contain infinitely many points, but this fact does not follow from the above axioms. Additional axioms will be needed to assure infinitude of points on a line.

## Notation for Points, Lines, and Planes

It is customary to designate points by capital letters of the Latin alphabet. Axiom A.2 assures that a line can be named by any two points on it. The line containing points  $A$  and  $B$  as in Figure 1 will be denoted by  $\overleftrightarrow{AB}$  or “line  $AB$ ”. Whenever convenient, we also may name a line by a single letter. In this text only lower case letters from the Latin alphabet will be used.

Axiom A.3 assures that a plane can be named by any three noncollinear points on the plane. For example, the plane containing the upper face  $ABCD$  of the box in Figure 6 can be named in each of the following ways: plane  $ABC$ , plane  $BCD$ , plane  $ADC$ , and plane  $ABD$ . (Of course the plane can be named by three other noncollinear points not labeled in the figure.) Whenever convenient we may also name a plane by a single letter. In this text only lower case Greek letters will be used to name planes. Recall that the



**Figure 6**

intersection of two sets is the set of all elements that are common to both sets. We know intuitively that two lines either do not intersect (have no points in common) or intersect in exactly one point. This is our first theorem.

**Theorem A.1.** *If two distinct lines intersect, they intersect in exactly one point.*

*Proof.* Let the lines be  $k$  and  $\ell$ . It is given that the lines intersect. Therefore, there exists a point  $P$  which is on both lines. We want to show that  $k$  and  $\ell$  have no other points in common. For that purpose we use an indirect proof. Suppose there is another point  $Q$  on  $k$  and  $\ell$ , as in Figure 7. By Axiom A.2, there is a unique line through  $P$  and  $Q$ . Hence,  $k = \ell$  which contradicts the hypothesis that the lines are different. Consequently, the existence of another intersection point  $Q$  must be rejected.  $\square$

Axiom A.3 assured us that three noncollinear points determine a plane. Are there other ways to uniquely determine a plane? The reader is probably aware of another way

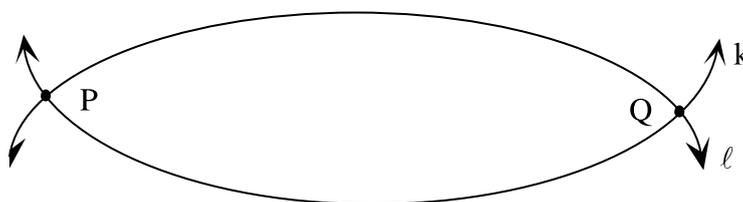


Figure 7

to determine a plane: *two parallel lines determine a plane*. However, we need first to define parallel lines.

#### DEFINITION OF PARALLEL AND SKEW LINES

Two lines are *parallel* if they lie in the same plane and do not intersect. Lines that do not intersect and are not contained in any single plane are called *skew* lines.

If  $\ell$  and  $m$  are parallel we write  $\ell \parallel m$ . Notice that in Figure 4  $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$ ,  $\overleftrightarrow{AB} \parallel \overleftrightarrow{EH}$ ,  $\overleftrightarrow{AC} \parallel \overleftrightarrow{FH}$ , but  $\overleftrightarrow{AB}$  and  $\overleftrightarrow{DE}$  are skew lines. (Later we will define a cube and will be able to prove these facts.)

#### INVESTIGATION 1

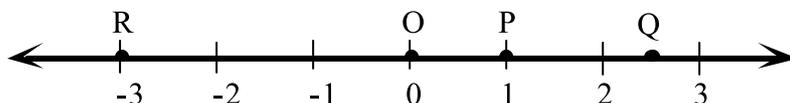
- What is the maximum number of intersection points determined by  $n$  lines in the same plane.
- What is the maximum number of lines determined by  $n$  points? Does it matter if the points are in the same plane or not?

### Intuitive Background for Coordinate System and Distance

Following G.D. Birkhoff's axioms (see Section 1.1), we introduce now the concept of distance, assuming the existence and properties of real numbers. We start with an intuitive background which will motivate the axioms and definitions that follow.

Given any line, a **coordinate system** on the line can be created by choosing arbitrary point  $O$  on the line and corresponding to it the number 0. The point  $O$  is called the **origin**. Then to the right of point  $O$  another point  $P$  is chosen (see Figure 8) to which

we correspond the number 1. The segment  $\overline{OP}$  is called a **unit** segment. By laying off



**Figure 8**

$\overline{OP}$  to the right and left of  $O$ , we find points corresponding to integers. By dividing segments into an appropriate number of equal parts we can find points that correspond to all rational numbers. The reader most likely knows that any real number (and not only rational numbers) corresponds to some point on the line, and, conversely, every point on the line corresponds to some real number. (In Chapter 3 we will show how to find points that correspond to real numbers such as  $\sqrt{2}$  and  $\sqrt{5}$ .) Thus, there is one to one correspondence between the points on a line and the real numbers. Such a correspondence is called a *coordinate system* for a line. The number corresponding to a given point  $P$  is called the *coordinate* of  $P$ . Thus, the coordinate of  $Q$  in Figure 8 is 2.5.

If we denote the line in Figure 8 by  $x$  we write the coordinate of  $Q$  as  $x_Q$ . Thus,  $x_Q = 2.5$  and  $x_R = -3$ . The distance between two points can be found using the coordinates of the points. For example, in Figure 8 we have  $PQ = OQ - OP = x_Q - x_P = 2.5 - 1 = 1.5$ . Notice that  $RP$  can also be found by finding the difference between the coordinates of the points:  $RP = x_P - x_R = 1 - (-3) = 4$ . Because distance is a nonnegative number and it is cumbersome to indicate which point has the greater coordinate, we use the absolute value function. Thus  $AB = |x_A - x_B|$ .

Based on this discussion we introduce the following axiom and definitions.

**Axiom A.7. The Ruler Postulate.** *The points on a line can be put in one to one correspondence with the real numbers.*

Notice that this axiom implies that every line has an infinite number of points.

**DEFINITION OF A COORDINATE SYSTEM FOR A LINE** The correspondence in Axiom A.7 is called a *coordinate system* for a line. A line with a coordinate system is called a *number line*.

**DEFINITION OF A DISTANCE BETWEEN TWO POINTS** The distance between points  $A$  and  $B$ , denoted by  $AB$ , is the real number  $|x_A - x_B|$  where  $x_A$  and  $x_B$  are the coordinates of  $A$  and  $B$ , respectively, in a coordinate system for  $\overline{AB}$ .

Notice that the distance between two points depends on the unit chosen for the coordinate system, that is, on the position of the point that corresponds to the number 1. Notice that because the distance from a point  $A$  to the origin is  $|x_A|$  and there exist real numbers as large as we wish, we conclude that there are points on a line as far from the origin as we wish. We say that a line is infinite in length.

The concept of distance can be used to define what is meant by saying that a point is between two other points.

**DEFINITION OF BETWEENNESS**  $B$  is between  $\overline{AC}$  if and only if  $A$ ,  $B$ , and  $C$  are collinear and  $AB + BC = AC$  (see Figure 7). In this case we write  $A - B - C$ .



Figure 9

Using the concept of betweenness for points it is possible to define various geometric figures.

**DEFINITION OF A SEGMENT** The **segment**  $\overline{AB}$  consists of the points  $A$  and  $B$  and all the points between  $A$  and  $B$ .

The length of segment  $v$  is the distance between  $A$  and  $B$ , that is,  $AB$ . A point  $M$  between  $A$  and  $B$  such that  $AM = MB$  is a **midpoint** of  $\overline{AB}$ . Our intuition tells us that every segment has exactly one midpoint. This fact can be proved using our axiom and definitions. You will be guided in the process of finding a proof in the problem set. Most of us probably find it more satisfying to prove statements that are not intuitively obvious. Nevertheless, you may find the challenge of proving a statement rewarding in itself even

if it seems obvious. We also want you to realize that even intuitively obvious statements can be logically deduced from the axioms and definitions.

**EXAMPLE A.1.** Given two points  $A$  and  $B$  on a number line with respective coordinates  $x_A$  and  $x_B$ , find the coordinate  $x_M$  of  $M$  the midpoint of  $\overline{AB}$ .

**Solution.** Assume  $x_B > x_A$ . The definition of the midpoint implies that  $AM = MB$ . This equation implies:

$$\begin{aligned}x_M - x_A &= x_B - x_M \\2x_M &= x_A + x_B \\x_M &= (x_A + x_B)/2\end{aligned}$$

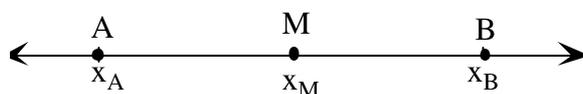


Figure 10

#### INVESTIGATION 2

A student approaches the solution of Example A.1 as follows: *Because  $AB = x_B - x_A$  and the midpoint of  $AB$  is halfway between  $A$  and  $B$ , the coordinate of the midpoint should be  $1/2(x_B - x_A)$ .* The student realizes the answer is wrong but would like to know why and how to use her approach to obtain the correct answer. How would you respond?

**DEFINITION OF A RAY** the ray  $\overrightarrow{AB}$  (shown in Figure 11(a)) is the union of  $\overline{AB}$  and the set of all points  $C$  such that  $B$  is between  $A$  and  $C$ . The point  $A$  is called the **endpoint** of the ray. The rays having a common endpoint and whose union is straight line are **opposite rays**. In Figure 11(b)  $\overrightarrow{AB}$  and  $\overrightarrow{BC}$  are opposite rays.

**DEFINITION OF AN ANGLE** An **angle** is a union of two rays with a common endpoint. The common endpoint is the **vertex** of the angle and the two rays are called the sides of the angle.

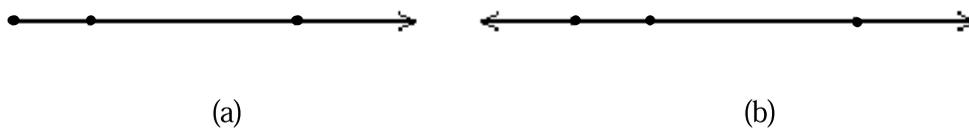


Figure 11

In Figure 12(a), the angle shown is the union of  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  and its vertex is  $A$ . (Using set notation, the angle is  $\overrightarrow{AB} \cup \overrightarrow{AC}$ .) The angle in Figure 12(a) is designated by  $\angle BAC$  or  $\angle CAB$ . When there is no danger of ambiguity, it is common to name an angle by its vertex. Thus, the angle in Figure 12(a) can also be denoted by  $\angle A$ . If the rays  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  are on the same line, i.e., if  $A$ ,  $B$ , and  $C$  are collinear as in Figure 12(b)  $\angle ABC$  is called a **straight angle**. If  $A$ ,  $B$ , and  $C$  are three noncollinear points, then the union of

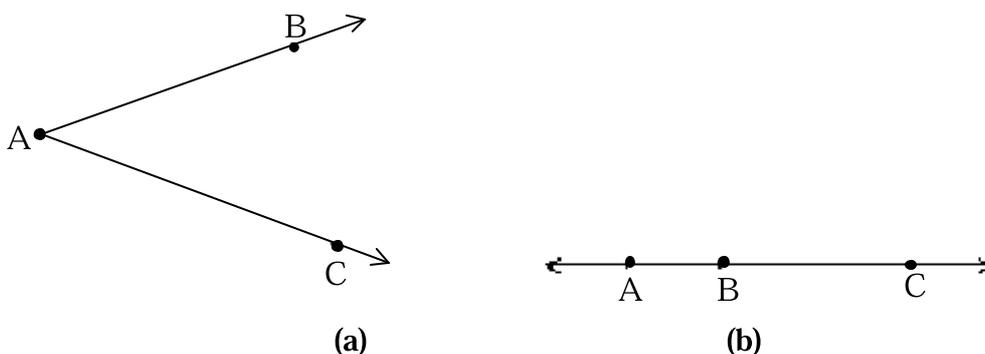


Figure 12

the three segments  $\overline{AB}$ ,  $\overline{BC}$ , and  $\overline{AC}$  is called a triangle and denoted by  $\triangle ABC$  (shown in Figure 13). The three segments are called the sides of the triangle and the three points the **vertices**.

At this point we could prove several statements which are intuitively obvious. For example, if  $A - B - C$  then  $C - B - A$ . Also  $AB \cong BA$  and  $\triangle ABC \cong \triangle CBA$ . By equal we mean “exactly” equal. That is in the set theory sense: *two sets are equal if they contain the same elements*. A rigorous treatment of subsequent topics requires the concept and properties of *half planes*. Intuitively, we know that any line divides the plane into two parts separated by the line and each is referred to as the *half plane*. This fact will be introduced in Axiom A.8, a *Plane Separation Axiom*. But before, it is handy to

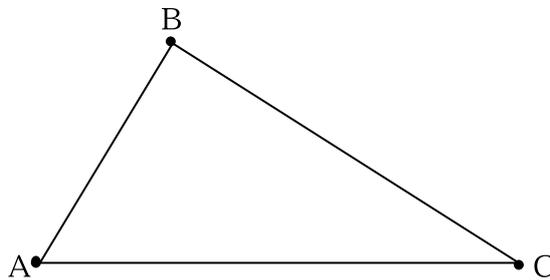


Figure 13

define what is meant by a *convex set*.

**DEFINITION OF A CONVEX SET** A set is convex if for every two points  $P$  and  $Q$  belonging to the set, the entire segment  $\overline{PQ}$  is in the set.

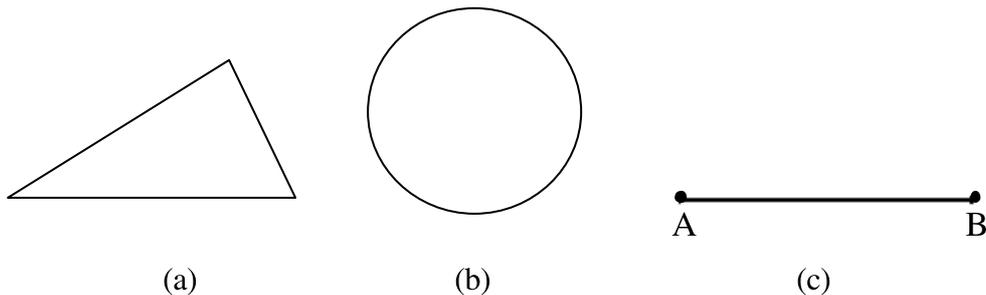


Figure 14

Notice that the interiors of the triangle as well as the circle in Figure 14(a) and (b) are convex sets. (We are making this statement on an intuitive basis. The interior of a triangle has not been defined yet.) Also the segment  $\overline{AB}$  is a convex set (why?). However, the interiors of the figures in Figure 15(a) and (b) are not convex, as segments  $\overline{PQ}$  in each figure does not entirely belong to the figure.

**PROBLEMS SOLVED AND UNSOLVED** Properties of convex sets have been extensively investigated and the concept has important applications in mathematics.

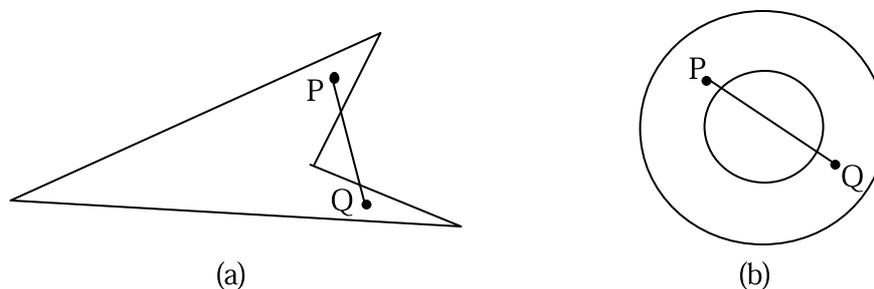


Figure 15

We briefly describe here two properties of convex sets. One of which has been proved and the other which has not. We use terminology that should be intuitively clear but has not yet been precisely defined.

**Chords intersecting at  $60^\circ$**  In the interior of any closed convex curve (See Figure 18) there exists a point  $P$  and three chords through  $P$  (a chord of a region is a segment connecting any two boundary points of the region) with the following property: The six angles formed at  $P$  are  $60^\circ$  each and  $P$  is the midpoint of each of the chords. (The proof of this statement requires more extensive study of a convex sets.)

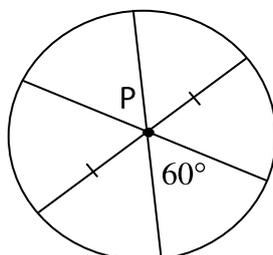


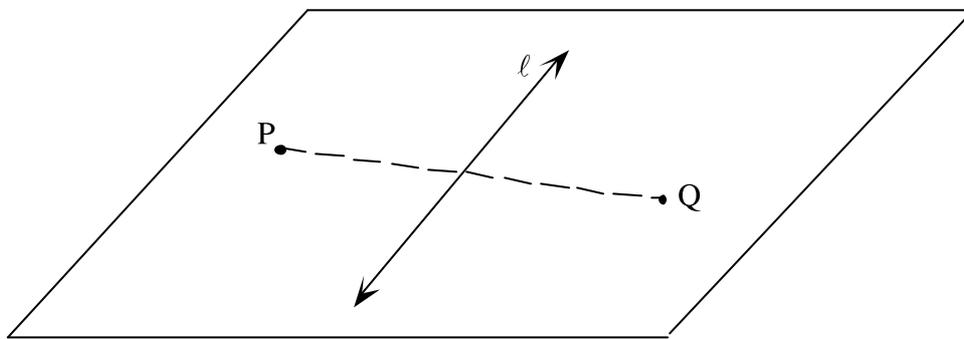
Figure 16

**Equichordal Points** A point  $P$  in the interior of a region is an equichordal point if all chords through  $P$  are of the same length. For example the center of a circle is an equichordal point of the circle. However, there exists non-circular regions that have an equichordal point. In 1916 Fujiwara raised the question of whether there exists a plane convex region which has two equichordal points. No one has been able to give

a complete answer to this question. In 1984 Spaltenstein described a construction of a convex region on a sphere which has two equichordal points. (This does not answer the question posed in 1916 as the region is on a sphere and hence not a planar.)

**Axiom A.8. The Plane-Separation Axiom.** *Each line in a plane separates all the points of the plane which are not on the line into two nonempty sets, called the half planes, with the following properties:*

- (i) *The half planes are disjoint (have no points in common) convex sets.*
- (ii) *If  $P$  is in one half plane and  $Q$  is in the other, the segment  $\overline{PQ}$  intersects  $\ell$ .*



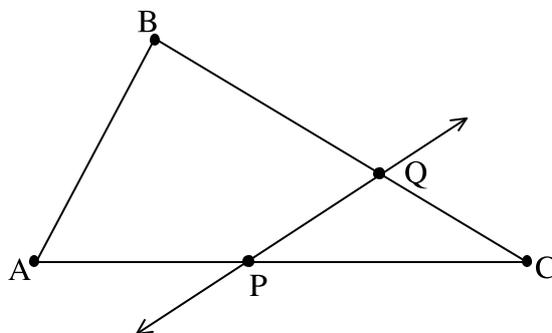
**Figure 17**

Notice that neither of the half planes in Axiom A.8 includes the line. Thus, a line divides the plane into three mutually disjoint subsets: the two half planes and the line. Also, it follows from Axiom A.8 that a half plane is determined by a line and a point not on the line. Thus we can refer to the two half planes in Figure 17 as *the half plane of  $\ell$  containing  $P$*  and *the half plane of  $\ell$  containing  $Q$* .

Using Axiom A.8 it is possible to prove a theorem named after the German mathematician Moritz Pasch (1843-1930) which states that if a line intersects one side of a triangle and does not go through any of its vertices it must also intersect another side (see Figure 18). In 1882, Pasch published one of the first rigorous treatises on geometry where he stated the theorem as an axiom (he did not use Axiom A.8 as an axiom). Pasch realized that Euclid often relied on assumptions made visually from diagrams and contributed to filling the gaps in Euclid's reasoning.

We state Pasch's Axiom as a theorem and leave its proof to be explored in the problem set.

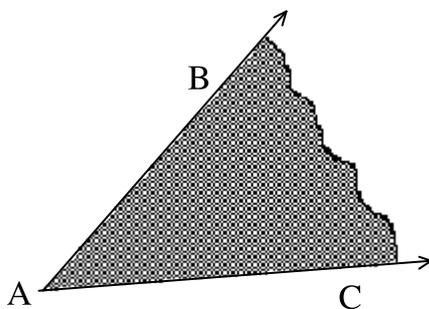
**Theorem A.2. Pasch's Axiom.** *If a line intersects a side of a triangle and does not intersect any of the vertices, it also intersects another side of the triangle.*



**Figure 18**

Using Axiom A.8 it is possible to precisely define the interior of an angle and hence the interior of a triangle.

**DEFINITION OF INTERIOR AN ANGLE** If  $A, B, C$  are not collinear then the interior of  $\angle BAC$  is the intersection of the half plane of  $\overleftrightarrow{AB}$  containing  $C$  with the half plane of  $\overleftrightarrow{AC}$  containing  $B$ . (See Figure 19.)



**Figure 19**

**Remark.** The above definition is valid (well defined) if it is independent of the choice of  $B$  and  $C$  on the sides of the angle. It can be proved that this indeed is the case.

Using the above definition it is possible to define the *interior of a triangle*. Coming up with an appropriate definition is left as an exercise. The definition of the *interior of an angle* can also be used to define betweenness for rays.

**DEFINITION OF BETWEENNESS OF RAYS**  $\overrightarrow{AD}$  is between  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  and if and only if  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  are not opposite rays and  $D$  is in the interior of  $\angle BAC$ . (See Figure 20.)

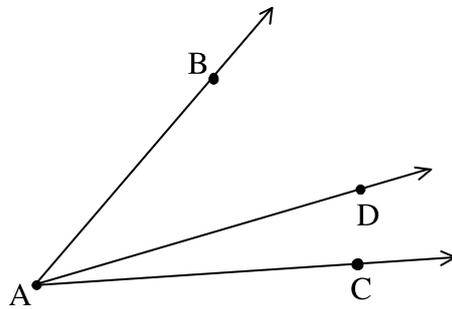


Figure 20

The above definition can be used to prove the following visually obvious theorem. (Its proof is left as an exercise.)

**Theorem A.3.**  $\overrightarrow{CD}$  intersects side  $\overline{AB}$  of  $\triangle ABC$  between  $A$  and  $B$  if and only if  $\overrightarrow{CD}$  is between  $\overrightarrow{CA}$  and  $\overrightarrow{CB}$ . (See Figure 21)

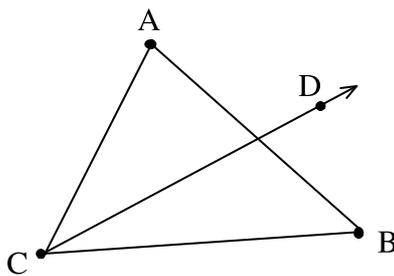


Figure 21

## Angle Measurement

Angles are commonly measured in degrees with a protractor. We will later introduce the radian measure as well. To measure  $\angle EAB$  we place the protractor as in Figure 22 and read off the measure of the angle as  $140^\circ$ . The following axioms and definitions formalize

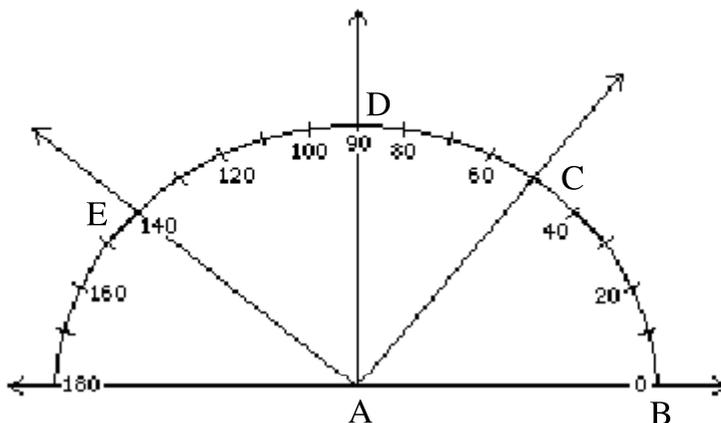


Figure 22

the intuitive knowledge of the protractor.

**Axiom A.9. The Angle Measurement Axiom.** *To every angle  $\angle BAC$  corresponds a real number between 0 and 180. To a straight angle corresponds the number 180.*

The number in Axiom A.9 is called the **degree measure of the angle** and is written  $m(\angle BAC)$ . In Figure 23  $m(\angle BAC) = 50^\circ$  and we say that  $\angle BAC$  is a 50 degree angle written  $50^\circ$ .

**Axiom A.10. The Angle Construction Postulate.** *Let  $\overrightarrow{AB}$  be a ray on the edge of a half-plane. For every real number  $r$ ,  $0 < r < 180$ , there is exactly one ray, with  $C$  in the half plane, such that  $m(\angle CAB) = r$ .*

**Axiom A.11. The Angle Addition Postulate.** *If  $D$  is a point in the interior of  $\angle BAC$  then  $m(\angle BAD) + m(\angle DAC) = m(\angle BAC)$ .*

Notice that Axiom A.11 implies that measures of angles can be computed by subtraction, e.g.,  $m(\angle BAD) = m(\angle BAC) - m(\angle DAC)$ .

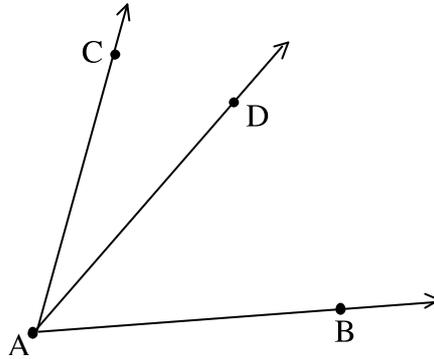


Figure 23

### Problem Set 1.1

In the following problems use the axioms, definitions and theorems of this section to prove what is required.

1. Prove that:
  - (a) If two lines intersect, they lie in exactly one plane.
  - (b) Two parallel lines determine a unique plane.
2. Consider the undefined terms: *ball*, *player*, and *belongs to* along with the following axioms:

**Axiom A.2** : There is at least one player.

**Axiom A.3** : To every player belong two balls.

**Axiom A.4** : Every ball belongs to three players.

Prove or disprove each of the following:

- (a) There are at least 2 balls.
- (b) There are at least 3 players.
- (c) There is always an even number of balls.
- (d) The number of players is always odd.

(Hint: You may want to model the balls by a point and players by segments.)

3. Consider the undefined terms: *line*, *point*, and *belongs to* along with the following axioms:

**Axiom A.2** : Any two lines have exactly one point in common.

**Axiom A.3** : Every point is on (belongs to) exactly two lines.

**Axiom A.4** : There are exactly 4 lines.

Answer each of the following:

- (a) How many points are there? Prove your answer.
- (b) Prove that there are exactly 3 points on each line.

(You may prefer to substitute the term *point* by *person* and *line* by committee.)

4. Prove Pasch's Axiom.
5. Using the terminology of this section come up with a precise definition of the interior of a triangle.
6. Is the *betweenness* of rays well-defined? Explain.
7. Which of the following is true *always*, *sometimes* but not *always*, or *never*? Justify your answers.
- (a) The intersection of two convex sets is convex.
  - (b) The intersection of two nonconvex sets is a nonconvex set.
  - (c) The union of two convex sets is convex.
  - (d) The union of two nonconvex sets is nonconvex.